Stability of Finite-Amplitude Autooscillations in Poiseuille Flow

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The conditions of existence of analytical invariant manifolds for Navier-Stokes equations are derived. A method of constructing the invariant manifolds in specific problems is suggested. The bifurcations of subcritical autooscillating regimes in a plane Poiseuille flow are studied.

Consider the flow of a viscous incompressible fluid in a bounded domain Ω with piecewise-smooth boundary $\partial\Omega$. The flow velocity $V(x,t)$ and pressure $p(x,t)$ are determined by a system of Navier-Stokes equations

$$\frac{\partial V}{\partial t} + V \cdot \nabla V = -\nabla p + \nu \nabla v + f ; \nabla \cdot \mathrm{V} = 0 \qquad (1)$$

Assume that at the given boundary conditions there exists a stationary solution of the system (1) (V_0, p_0)and we shall search for the solution of the form

$$V(x,t) = V_0(x) + u(x,t), \quad p(x,t) = p_0(x) = q(x,t)$$

Introduce the Hilbert spaces

$$H = \{u \varepsilon [L_2(\Omega)]^3 ;\ \nabla \cdot u = 0 ;\ u \cdot n/_{\partial\Omega} = 0\}$$

$$K = \{ u \varepsilon [H^1(\Omega)]^3 ;\ \nabla \cdot u = 0 ;\ u \cdot n /_{\partial\Omega} = 0\}$$

$$D = \{ u \varepsilon [H^2(\Omega)]^3 ;\ \nabla \cdot u = 0 ;\ u \cdot n /_{\partial\Omega} = 0\}$$

where $H^s(\Omega$ is the Sobolev's space.

Then (see, f.e. [1]) a problem of finding (u,q) reduces to a differential equation with unbounded operators in the Hilbert space H

$$\frac{du}{dt} = -L_\nu u + N(V;u) \qquad (2)$$

where L_ν is the linear operator with the domain D

$$L_\nu = \Pi(\nu\, \Delta u - V_0 \cdot \nabla u - u \cdot \nabla V_0)$$

(Π is the orthogonal projector on H in $[L_2(\Omega)]^3$.

The nonlinear operator N has the domain K and in the present case does not depend on the parameter v

$$N(u) = -\Pi u \cdot \nabla u$$

Consider a general case of Eq.(2) in some Hilbert space. Assume that the linear L_v and nonlinear N operators are closed and satisfy the following conditions:

I. $N(V;0) = 0$, $v \varepsilon R^m$, $N_u(v;0) = 0$.

II. A domain of the operator L_v $-D(L)$ does not depend on v.

III. The operator L_v is m-sectorial. Hence it follows that $(-L_v)$ is the generating operator of the analytical semigroup and that there exists the operator $(-L_v - a_v)^\alpha$, $0 < \alpha < 1$, $D[(-L_v - a_v)^\alpha] \supset D(L)$
$(a_v$ is the vertex of sector inside which the eigenvalues of the operator L_v lie).

IV. The operator $N(v, x)$ has a domain $D(N) \supset D(L)$ independent of v and $D(L)$ is the mapping of $D(L)$ in $D(-L_v - a_v)$ at some α. This mapping possesses the Lipschitz-continuous first derivative.

V. A spectrum of the $(-L_v)$ - δ permits the partition $\delta = \delta_1 \cup \delta_2$ where δ_1 is a bounded portion of δ.

VI. $æ_v > 0$, $æ_v > q_v$

where $æ_v = -\sup_{ϱ \varepsilon \delta_2} Re\, ϱ$, $q_v = -\inf_{ϱ \varepsilon \delta_1} Re\, ϱ$

We shall study classical solutions of the initial problem for Eq.(2)

$$u(0) = u_0 \varepsilon D(L) \ ; \quad u \,\varepsilon\, C^0(0, \infty ; D(L) \cap C^1 \infty ; H) \qquad (3)$$

By virtue of the condition V there exists the decomposition of the operator L_v, with respect to a direct sum of the orthogonal subspaces $H = P_v H \oplus (I - P_v) H$ where P_v is such a projector that the spectrum of the operator $(-P_v L_V)$ equals δ_1. Any vector $u \,\varepsilon H$ can be represented in the form of sum $u = y + z$, $y \,\varepsilon P_v H$, $z \varepsilon (I - P_v) H$. In what follows a notion of invariant manifold vail be of importance. We shall call an invariant manifold of Eq.(2) a set $M \subset H$ of the form

$M = \{y, \tilde{z}(y)\}$, $y \,\varepsilon P_v D(L)$, $\tilde{z}(y) \,\varepsilon (I - P_v) D(L)$

such that if $u_0 \varepsilon M$, then the whole trajectory $u(t)$, $0 \le t < \infty$ belongs to M.

Then there holds the following

Theorem 1. Let the conditions I - VI be satisfied. Then there exists such a number $\beta > 0$ that the following statements hold:

1. Eq.(2) has an invariant manifold

$$M = \{y, \tilde{z}(y)\}, \ \ \tilde{z}(0) = 0\ , \ \ \|y\|_d \le \rho$$

$$\rho + \rho^{1/\alpha} < \beta\,(æ_v - q_v)$$

The mapping $\tilde{z}$; $P_v D(L) \to (I - P_v) D(L)$ satisfies the Lipschitz condition and $\|\tilde{z}\|_d \le \rho$

$$(\|x\|_d = \|x\| + \|L_v x\|)$$

2. Any global solution of the problem (2),(3) with the property

$\|P_v u(t)\|_d \le \rho$, $t \varepsilon [0, \infty)$

is attracted by the manifold M, i.e. there may be found such a number $\gamma > 0$ that

$$\|\tilde{z}(P_v u(t)) - (I - P_v) u(t)\|_d \le const \|\tilde{z}(y_0) - z_0\|_d\, \ell^{-\gamma t}, \quad t \to \infty$$

where

$$y_0 = P_v u_0\ , \ z_0 = (I - P_v) u_0$$

From the theorem I it follows that the study of trajectories of the evolutionary equation (2) with unbounded operator may be reduced to the analysis of solutions of the evolutionary equations with bounded operators $L_I = P_v L_v$ and $N_1 = P_v N$. If the spectrum δ_I consists of a finite number of isolated eigenvalues, then the problem reduces to the study of the finite-dimensional dynamic system. For finding the manifold of M we shall formulate the additional conditions on the operators L_v and N.

VII. The nonlinear operator $N\,(v; x)$ is an analytical operator acting from $\{v;\ D(N)$ in $H.$

VIII. The vector $L_v\ x$ depends analytically on v at any $x \varepsilon D(L)$

IX. The spectrum δ_I consists of n pairs of simple isolated eigenvalues $\{\lambda_i, \tilde{\lambda}\}$

$$\lambda_i(v) = \gamma_i(v) + i w_i(v)\ , \ w_i(v) > 0\ , \ \ i = 1, 2, \ldots, n$$

If at some natural $k > 0$ there holds the equality

$$k\, w_1(v_0) = w_{i_0}(v_0)\ , \quad v_0 \varepsilon R^m$$

then $\quad \gamma_{i_0}(v_0) \ne 0$

the conjugate operator $(-L_v^*)$. Let $p^{(i)}$ be a projector on the eigenspace $(-L_v)$ answering a pair of eigenvalues $\lambda_i\ \overline{\lambda}_i$. Then

$$P^{(i)} x = (x, \psi_i + (x, \overline{\psi}_i) \overline{\varphi}_i \equiv y_i\ , \quad (\varphi_I, \ \ \psi_i) = 1$$

The equation (2) is equivalent to the following system

$$\frac{dy_i}{dt} = -P^{(i)}L_v y_i + P^{(i)}N(v, y_1 + y_2 + ... + y_n + z)\,; \quad i = 1, 2, \ldots, n$$

$$\frac{dz}{dt} = QL_v z + QN(v; y_1 + y_2 + ... + y_n + z)\,; \quad z\,\varepsilon\, QD(L) \tag{4}$$

$$Q = I - \sum_{i=1}^{n} P^i$$

Introduce in the subspaces $P^{(i)}H$ the systems of polar coordinates and we shall search for the solutions of Eq.(4) in the form

$$y_i = R\ell(r_i \ell^{i\theta i}\varphi_i) + \widetilde{y}_i(r_1, ...r_n; \theta_1, ...\theta_n)$$

$$\int_0^{2\pi} \ell^{-i\theta i}(\widetilde{y}_i, \psi_i)d\theta_i = 0 \tag{5}$$

$$z = \widetilde{z}(r_1, ..., r_n \quad ; \quad \theta_1, ..., \theta_n)$$

Substituting (5) into (4), we can derive a finite-dimensional autonomous system of equations for r_i, θ_i

$$\frac{dr_i}{dt} = (\gamma_i + b^{(i)})r_i \qquad \frac{d\theta_i}{dt} = w_i + c^{(i)} \quad ; \quad i = 1, 2, \ldots n \tag{6}$$

$$b^{(i)} = R\,\ell << N,\ \psi_i >>\ ,\ c^{(i)} = I_m << N,\ \psi_i >> = \frac{1}{2\pi r_i}\int_0^{2\pi} \ell^{-i\theta i}(N, \psi_i)d\theta_i$$

In what follows we shall construct the invariant manifold from integral curves of Eq.(2). For this purpose we make use of the first integrals of the system (6) which establish the relationship between two points of the integral curve of this system

$$(r^{(1)}, \theta^{(1)})\ ,\ (r^{(2)}, \theta^{(2)})$$

$$\begin{aligned} r_i^{(2)} &= \hat{r}_i(\theta_1^{(2)}; r_1^{(1)}, \ldots, r_n^{(1)}, \theta_1^{(1)}, \ldots, \theta_n^{(1)})\,, \quad i = 1, 2, \ldots, n \\ \theta_i^{(2)} &= \hat{\theta}_i(\theta_1^{(2)}; r_1^{(1)}, \ldots, r_n^{(1)}, \theta_1^{(1)}, \ldots, \theta_n^{(1)})\,, \quad i = 2, \ldots, n \end{aligned} \tag{7}$$

If $w_1 + c^{(1)} \neq 0$, then the functions $\hat{r}_i, \hat{\theta}_i$ are found from equations

$$\begin{aligned} \frac{d\hat{r}_i}{d\theta_1} &= \frac{\gamma_i + b^{(i)}\hat{r}_i}{w_1 + c^{(1)}} \quad ; \qquad i = 1, 2, \ldots, n \\ \frac{d\hat{\theta}_i}{d\theta_1} &= \frac{w_i + c^{(i)}}{w_1 + c^{(1)}} \quad ; \qquad i = 2, \ldots, n \end{aligned} \tag{8}$$

The equation

$$\frac{d\theta_1}{dt} = w_1 + c^{(1)} \tag{9}$$

determines the dependence of solution of the system (8) on time. When deriving the equations for $\widetilde{z}$ and $\widetilde{y}$ we shall assume that in formulas (7) the point $(r^{(2)}, \theta^{(2)})$ is fixed and the point ($r^{(1)}$, $\theta^{(2)}$) changes along the Integral curve. As a result we obtain

$$w_1 \frac{\partial \tilde{y}_i}{\partial \theta_1} + P^{(i)} L_v \tilde{y}_i = -c^{(1)} \frac{\partial \tilde{y}_i}{\partial \theta_1} + P^{(i)} N - R\ell\,(<<N\psi_I>> r_i e^{i\theta} \varphi_I)$$

$$w_1 \frac{\partial \tilde{z}_i}{\partial \theta_1} + Q L_v \tilde{z}_i = -c^{(1)} \frac{\partial \tilde{z}_i}{\partial \theta_1} + QN \qquad (10)$$

Theorem 2. Let the conditions I-IX be satisfied. Then there may be found such $\rho > 0$ that at $|r| < \rho_0$ ($|r| = r_1 + \ldots + r_n$)the equation (1) has a unique 2n-dimens±onal invariant manifold M_{2n}

$$M_{2n} = \{R\,\ell(r_1 e^{i\theta_1} \varphi_1) + \bar{y}_1(r;\theta), \ldots, R\ell((r_n e^{i\theta_n} \varphi_n) + \tilde{y}_n(r;\theta),\ \tilde{z}(r;\theta)\ \}$$

The functions $\tilde{z}(r;\theta)$, $y_n(r;\theta)$ satisfy the equations (10), analytically depend on $r_i\,(i=1,\ldots,n)$ and on the parameter ν and are continuously differentiable 2π- periodical functions of $\theta_i\ (i=1,\ldots,n)$. The phase trajectories of Eq.(2) on the manifold are determined by the system (8) where $b^{(i)}\ (r;\theta_I,\ldots\theta_{i-1},\theta_{i+1},\theta_n)$,
$c^{(i)}(r;\theta_I,\ldots\theta_{i-1},\theta_{i+1},\theta_n$ are analytical functions of r, and ν and are continuously differentiable, 2π-periodical functions of θ_i. There may be found such $\beta_n > 0$ that at $|r| + |r|^{1/\alpha} < \beta_n(æ_\nu - q_\nu)$ the manifold M_{2n} is attracting. From the Theorem 2 it follows that the solution of Eq,(2) on the manifold M_{2n} has the form

$$u = \sum_{i=1}^{n} R\ell(r_i e^{i\theta_i} \varphi_i) + \tilde{\tilde{z}}\ ; \quad \tilde{\tilde{z}} = \sum_{i=1}^{n} \tilde{y}_i + \tilde{z}\ ; \quad \tilde{\tilde{z}} = \sum_{iSj=2}^{\infty} z_S r^S \qquad (11)$$

$$r^S = r_1^{S_1} \ldots r_n^{S_n}$$

Assuming also

$$b^{(i)} = \sum_{iSj=1}^{\infty} b_s^{(i)} r^S \quad , \quad c^{(i)} = \sum_{iSj=1}^{\infty} c_s^{(i)} r^S \qquad (12)$$

we obtain a recurrent system of linear equations for z_S

$$w_1 \frac{\partial z_S}{\partial \theta_1} + L_v z_S = -R\ell \sum_{i=1}^{n} (b_{s_i}^{(i)} + c_{s_i}^{(i)}) e^{i\theta_i} \varphi_i - \sum_{k+p=S} c_k^{(1)} \frac{\partial zp}{\partial \theta_1} + N_S \qquad (13)$$

The periodical functions $b_s^{(i)}$, $c_S^{(i)}$ are unambiguously determined by the condition

$$\int_0^{2\pi} e^{-i\theta_i} (z_s, \psi_i)\, d\theta_i = 0$$

It is possible to show that for Navier-Stokes equations in a bounded domain the conditions I-VIII are satisfied, for α in the condition IV the estimate $\alpha < 1/4$ is valid. In the case of spatial periodicity of the flow we can take a periodicity cell as Ω . The implementation of the condition IX depends on a specific problem. For example, it was shown in [2] that for a plane Poiseuille flow the neutral eigenvalue is almost everywhere simple on the neutral curve.

Consider the two-dimensional invariant manifold for disturbances of a plane Poiseuille flow. Take a pair of the first eigenvalues of the corresponding Orr-Sommerfeld problem $(\lambda_1, \bar{\lambda}_1)$ (the ordering is carried out with respect to γ_i) as δ_I. It is easy to show that the coefficients in expansions (12) are numbers and different from zero are only the coefficients with even indices. For numerical computations it is convenient to go over to the equation for stream function. Then the system (13) reduces to a system, of boundary-value problems for ordinary differential equations for the determination of harmonics of functions z_s - z_{sk}, $|k| \le S$. The fixed points of the equation for the phase trajectories

$$\frac{dr}{d\theta} = \frac{(\gamma_1 + b(r)r}{w_1 + c(r)}$$

determine the amplitudes of limiting cycles. It is known that for the Poiseuille flow there takes place a subcritical bifurcation. In [3,4] it was shown on the basis of the asymptotic method that the amplitude curve in the (r, R) plane, R being the Reynolds number, has a special point - a turning point. Consider a subcritical domain of Reynolds numbers and disturbance wavenumber α values. Introduce the notations

$$v = (\alpha, R)\,;\ r^2 = p\,;\ \varrho_0 = -\ \gamma;\ \ \varrho = b_{2e}\,(\ell = 2m - 1);\quad \varrho_\ell = -\,b_{2\ell}\ (\ \ell = 2m), m = 1,2,\ldots$$

Write down the equation for amplitudes of limiting cycles in the form

$$-\,\varrho_0(v) + f_1\,(v, p)\,p - f_2\,(v, p)\,p^2 = 0 \qquad (14)$$

Where

$$f_1\,(v, p) = \varrho_1(v) + \sum_{k=1}^{\infty} \varrho_{2k+1}(v) p^{2k}\,;\ f_2\,(v, p) = \varrho_2(v) + \sum_{k=1}^{\infty} \varrho_{2k+2}(v)\,p^{2k}$$

From (14) we derive equations for two pieces of the amplitude surface

$$p = \frac{f_1(v,p) + \Phi^{1/2}(v,p)}{2 f_2(v,p)} \qquad (15)$$

$$p = \frac{f_1(v,p) - \Phi^{1/2}(v,p)}{2 f_2(v,p)} \;; \qquad \Phi = f_1^2 - e_0 f_2 \qquad (16)$$

At the fixed $\alpha = \alpha_0$ the amplitude curve has a turning point at $R = R_H$ where $\Phi = 0$. At small p

$$f_1 \approx b_2(\alpha, R), \quad f_2 \approx -b_4(\alpha, R), \qquad \Phi \approx b_2^2 - 4\gamma_1 b_4$$

In the paper the numerical calculation of coefficients b_2, b_4 and approximate values of R_H is carried out. Table I represents the results r_H is the critical value of the amplitude, R_0 is the linear neutral value of R.

Table I

α	1.10	1.08	1.06	1.04	1.02	1.0	0.98
R_H	4960	4896	4905	4968	5068	5197	5301
$r_H 10^2$	0.63	0.71	0.79	0.89	1.03	1.28	1.84
R_0	8417	6359	5945	5849	5774	5880	5898

The computations have shown that at $0.94 \le \alpha \le 0.96$ the coefficient b_4 changes its sign in the vicinity of the turning point. In this connection on the basis of equations (15),(16) we can show that the amplitude curve may have a second turning point. This possibility is depicted in Fig.1 by dotted line. Then in the (α, R) plane there may exist a point where two turning points mix together. The cusp of the amplitude surface depicted in Fig.2 corresponds to this point.
The calculation of four-dimensional invariant manifold with the accuracy up to the second order term with respect to r_1. r_2, has been carried out. The first eigenvalues for odd eigenvalues of Orr-Sommerfeld equations were taken as the second pair of eigenvalues for δ_1. Numerical computations of Eq.(8) have shown that at $\alpha = 1$, $R = 5000$ and $r_1 < 0.9 \cdot 10^{-2}$ the two-dimensional manifold is stable with respect to four-dimensional disturbances. In calculations the Poiseuille flow velocity profile was taken

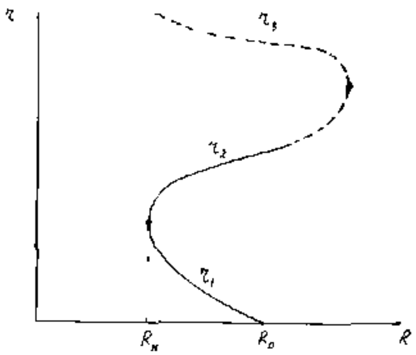

Fig.1. r_1, r_3 are unstable and r_2 stable limit cycles

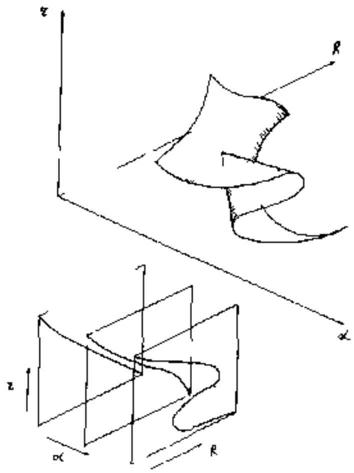

Fig.2

in the form $V_0 = y(2-y)$. The eigenfunctions of the Orr-Sommerfeld equation $\varphi(y)$ were normalized by conditions $\varphi(1) = 1$ for even functions and $\varphi'(1) = 1$ for odd ones. The integration of equations was carried out by the orthogonalization method. Each equation was integrated on its nonuniform mesh which condensed in a region of critical layer and near wall.

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