The Study of Shear Layer Stability by the Method of Vortex Particles

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Summary

The questions of the stability of the shear layers with respect to disturbances of its boundary and to the acoustic disturbance are examined. The problem is solved by the method of discrete vortex particles. Linear stage of development of the disturbances has been investigated in great detail, analytically as well as numerically, then nonlinear stage is investigated numerically only.

The problem of stability of hydrodynamic flows is still of interest despite a numerous literature. It is connected with the fact that up to now we are not able to study a nonlinear stage of the disturbance development and have a poor understanding of the transition from a linear stage to a nonlinear one. By solving problems on the stability some disturbance is always introduced, the nature of which is not discussed. At the same time it seems to us that for elucidating the mechanisms of the instability development it is necessary to study the influence of specific types of disturbances typical of the aerodynamic flows. In particular, in our opinion the study of acoustic disturbances is of special interest. There exists a numerous literature devoted to studying the influence of acoustics on the flows as well as the opposite influence of the flow on acoustics (see, f.e., reviews [1,2]). However, usually the boundary value problems are studied in which there take place the interactions of acoustics with a boundary as well as with the flow and boundary inhomogeneities and other complex interference phenomena. For this reason we have chosen for examination an infinite problem on the vortex layer stability under the action of acoustic disturbances. A two-dimensional (in the x,y plane) motion of unbounded compressible fluid is considered. The flow with the velocity components $(U,0)$ in the region $y<0$ and $(-U,0)$ in the region $y<0$ is the simplest flow, the stability of which will "be discussed. Thus, the question is about the stability of the vortex sheet. The stability of the vortex layer of a finite thickness Δ with a constant distribution of vorticity and of the layer with a profile of undisturbed velocity $V_x = Uth(y/\Delta)$ is also under consideration. We have also investigated the problem of the stability of the shear layers with respect to disturbances of its boundary. Let a plane acoustic wave propagate in a region y < 0. In a general case this wave undergoes the reflection and refraction when passing through a vortex layer. We shall consider a case of small Mach numbers $M = U/c_0 << 1$. Then the mentioned effects may be neglected in the zero approximation.

The problem is solved by the method of discrete vortex particles. This method has been developed by the authors of [3,4] for the case of an incompressible fluid. Therefore when solving the formulated problem we have to modify the above method. While deriving the equations of motion of discrete vortex particles we proceed, similarly as it was done in [3,4], from a formula representing the velocity field $\vec{v}$ through the vorticity $\vec{w}$ which in Lagrangian variables for the two-dimensional isentropic flows has the form:

$$\frac{d\vec{r}}{dt} = -\frac{1}{2\pi}\int \frac{[\vec{r}(\vec{\xi},t)-\vec{r}(\vec{\xi},t)]\times\vec{w}_0(\vec{\xi}')}{\left|\vec{r}(\vec{\xi},t)-\vec{r}(\vec{\xi}',t)\right]^2} d\vec{\xi}' + \vec{\nabla}\varphi \qquad (1)$$

here $\vec{w}_0(\vec{\xi}')$ is the initial vorticity, $\vec{\nabla}\varphi$ is the potential part of the velocity field.

Unlike the incompressible fluid Eq.(1) is not closed, another equation is needed for determining the potential φ. To derive it we shall proceed from the impulse equation written in the Lamb form [5]. Calculating the divergence of both sides of this equation and using the discontinuity equation, we can derive an equation for determining the potential. In a general case this equation is rather complex. However, in the problem under study it is simplified as the amplitude of the potential part of the velocity field is small as compared with U and takes the form

$$\left(\frac{\partial}{\partial t}+\vec{u}\cdot\vec{\nabla}\right)^2\varphi=c_0^2\Delta\varphi+\left(\frac{\partial}{\partial t}+\vec{u}\cdot\vec{\nabla}\right)\left(\Delta^{-1}w^2-\frac{u^2}{2}\right) \tag{2}$$

where $\vec{u}$ is the solenoidal part of the velocity. For sufficiently slow

$$\frac{\partial^2\varphi}{\partial t^2}=\Delta\varphi+\frac{M}{\varepsilon}\frac{\partial}{\partial t}(\Delta^{-1}w^2-\frac{u^2}{2}),\quad \varepsilon=\frac{A}{\lambda U}, \tag{3}$$

here *A* and λ are the amplitude and the wavelength of acoustic disturbance, respectively.

Prom Eq.(3) it follows that the nonstationary vortex motions of fluid generate an aerodynamic sound even at uniform initial conditions. In what follows for simplicity we confine ourselves to the case when $M \ll \varepsilon$. At this condition we may neglect the aerodynamic sound generated by nonstationary motion of vortices that allows in a pure form to study the influence of external acoustic field on the fluid flow. Then in the zero approximation the potential φ satisfies the wave equation. In what follows we shall consider the interaction of the examined flow with a plane harmonic wave. In this case the second term in the right-hand side of Eq.(1) is determined by expression $\vec{\Delta}\varphi=-A\vec{k}\sin(\vec{k}\cdot\vec{r}(\vec{\xi},t)-\Omega t,$ ($\vec{k}$ is the wave vector and Ω is the frequency).

Carrying out similarly as in our works [3,4] the discretization of Eq.(1), we arrive at a system of equations for discrete vortex particles. In computations we made use of the Gauss form of distribution of the vortex particle

$$\frac{dr_\alpha}{dt}=-\frac{1}{2\pi}\sum_{\beta}^{N}\Gamma_\beta\left\{1-\exp[-\frac{\left|\vec{r}_\alpha-\vec{r}_\beta\right|^2}{\delta_\alpha^2+\delta_\beta^2}]\right\}\frac{(\vec{r}_\alpha-\vec{r}_\beta)\times\vec{n}}{\left|\vec{r}_\alpha-\vec{r}_\beta\right|^2}- \tag{4}$$

$$-A\vec{k}\sin(\vec{k}\cdot\vec{r}_\alpha-\Omega t),\qquad \alpha=1,2,\ldots..N$$

where Γ_α is the circulation of the vortex particle α, δ_α^2 is its dispersion, $\vec{n}$ is the unit vector normal to the plane(x, y). Thus,the solution of the problem on the shear layer stability with respect to acoustic disturbances reduces to the solution of the equations of motion of the vortex particles in the external field.

As our purpose is to examine both the linear and nonlinear stages of development of acoustic disturbances in shear layers, it is advisable to study the problem analytically before stating the results of numerical computations. We list here the results of solving the problem of the linear theory of stability for vortex sheet although the similar computations may be carried out also for a vortex layer of a finite length with linear distribution of velocity.

Thus let $h(x,t)$ be a function determining a shape of the vortex sheet. On

$$\frac{\partial h}{\partial t}+\frac{\partial \Phi_{\mp}}{\partial x}\frac{\partial h}{\partial x}=\frac{\partial \Phi_{\mp}}{\partial y}, \quad \frac{\partial \Phi_{-}}{\partial t}+\frac{1}{2}(\vec{\nabla}\Phi_{-})^2=\frac{\partial \Phi_{+}}{\partial t}+\frac{1}{2}(\vec{\nabla}\Phi_{+})^2, \qquad (5)$$

where Φ is the velocity field potential. Here and everywhere in what follows the values pertaining to a region $y<h$ are marked by the index minus and those pertaining to a region $y>h$ by the index plus. The last condition is a consequence of the pressure continuity on the vortex sheet. Now represent the potentials $\Phi_{\pm}$ in the form of the sum of a potential induced by the vortex sheet $\Psi_{\pm}$ and of a potential $\varphi_{\pm}$ conditioned by the acoustic field presence. We have shown that for the present problem $\varphi_{\pm}\equiv\varphi=A\cos(\vec{k}\vec{x}-\Omega t)$. The potentials $\Psi_{\pm}$ satisfy the Laplace equation in the corresponding regions and they may be represented in the form $\Psi_{\pm}=\pm U_x+\Psi_{\pm}'$ ($\Psi_{\pm}'$ are the deviations of the potentials $\Psi_{\pm}$ from the undisturbed values $\pm U_x$ Substituting these expressions into Eq.(5) and linearizing them, we find

$$\frac{\partial h}{\partial t}\pm U\frac{\partial h}{\partial x}=\frac{\partial \Psi_{\mp}'}{\partial y}+\frac{\partial \varphi}{\partial y}, \quad \Delta\Psi_{\pm}'=0,$$

$$\frac{\partial \Psi_{+}'}{\partial t}-\frac{\partial \Psi_{-}'}{\partial t}+U\left(\frac{\partial \Psi_{+}'}{\partial x}+\frac{\partial \Psi_{-}'}{\partial x}\right)=2U\frac{\partial \varphi}{\partial x} \qquad (6)$$

with boundary conditions $\Psi_{\mp}'=0$ at $y\to\mp\infty$.

A general solution of a system of equations (6) without source terms has been obtained by Helmholtz, A particular solution will be sought for in the following form $\left(C_x=\Omega/k_x\right)$

$$h=A_h\,e^{ik_x(x-c_xt)}, \quad \Psi'\mp=A_{\mp}(y)\,e^{ik_x(x-c_xt)},$$

Then it is easy to make sure of the fact that the functions $A\pm(y)$ and A_h

$$A_t(y)=a_{\pm}\exp(\mp k_xy), \qquad A_h=-\frac{AK_yc_x}{K_x(C_x^2+U^2)}-i\frac{AU}{C_x^2+U^2},$$

$$\alpha_{\mp} = \mp \frac{AU(c_x + U)}{c_x^2 + U^2} + \frac{Ak_y U(c_x - U)}{K_x(c_x^2 + U^2)} .$$

Now a general solution of a system of equations (6) may be obtained in a general way. In particular, for the function h the following expression

$$h(x,t) = -\frac{Ac_x k_y}{(c_x^2 + U^2)k_x}\cos(k_x x - \Omega t) + \frac{AU}{c_x^2 + U^2}\sin(k_x x - \Omega t) + \\ + (c_{11}e^{\delta t} + c_{12}e^{-\delta t})\cos(k_x x) + (c_{21}e^{\delta t} + c_{22}e^{-\delta t})\sin(k_x x), \delta = k_x U \qquad (7)$$

is derived.

Thus, the presence of acoustic field leads to the vortex sheet disturbance. This disturbance forms both from acoustic disturbance the amplitude of which is limited and moreover small and from vortex disturbance excited by acoustics whose amplitude increases exponentially in a general case. It is possible to show that this disturbance will be of sinusoidal character.

How consider the computational results of the vortex sheet stability with respect to acoustic disturbances. In the method suggested in the present paper the problem is reduced to solving a system of equations (4). In computations the number of vortex particles N was varied from 120 up to 336. The dispersions of the vortex particles δ_α were determined in a similar way as in [4].

The equations of motion (4) are valid for the description of two-dimensional flows of a compressible fluid when the following conditions $\mathrm{M} \ll 1$, $\varepsilon < 1$, $\varepsilon > \mathrm{M}$ are satisfied. Therefore the specific computations were carried out by the following values of the parameters $\mathrm{M} = 10^{-2}$, $\varepsilon = 10^{-1}$, $\theta = -\pi/4$ (θ is the angle between the vector $\vec{k}$ and the axis x). By such a choice of parameters the amplitudes of disturbances of the transverse velocity component generated by external acoustic field turn out to be of the order $10^{-3} \div 10^{-4}$ at the initial stage. The computations were carried out at the time interval equal to several units $\mathrm{T} = \lambda / U$.

A typical picture of the vortex sheet evolution in the presence of external acoustic field is shown in Fig.1 (N = 336). The

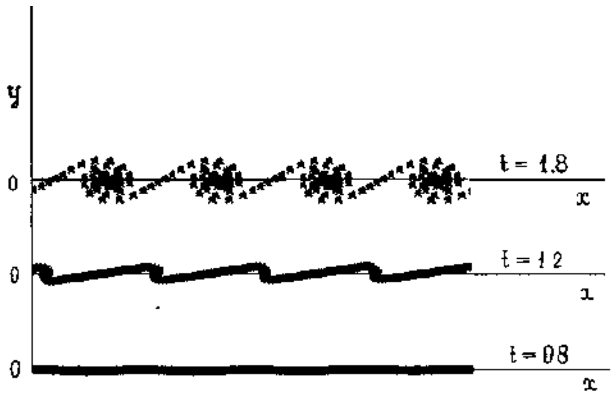
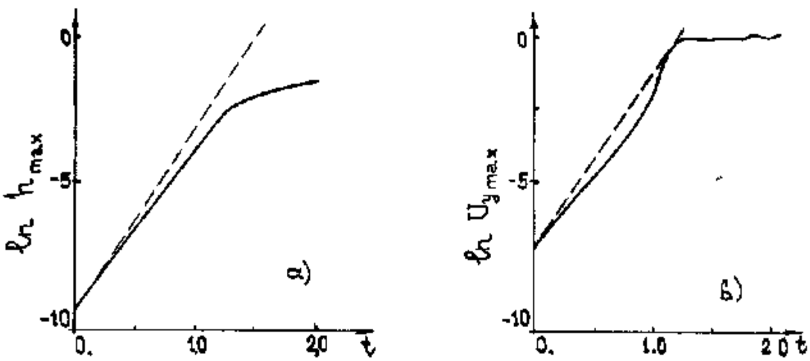


Fig.1. Evolution of the vortex sheet in the external acoustic field

initial location of the vortex sheet is rectilinear. At the first stage $t \leq 0.8$ the vortex sheet shape is sinusoidal and its amplitude and that of the transverse velocity component grow exponentially in time. When the amplitude of the vortex sheet achieves a value of the order 0.01 the nonlinear effects begin to be displayed and as a result a shape of the vortex sheet deviates more and more from sinusoidal. By the time moment $t = 1.2$ a shape of the vortex sheet becomes "sawtooth" and later we can observe its destruction that leads to the formation of large vortex structures.



Pig.2. Rate of the increase of the functions $\boldsymbol{h}_{max}$ and $\vartheta_{y\,max}$

Fig. 2 shows the graphs of the rate of the increase in the logarithm of the amplitude of disturbances of the transverse velocity component (Fig.2a) and of the logarithm of the function h amplitude (Fig. 2b). Here the dotted curves correspond to the linear theory. The analysis of these dependencies indicates that at the initial stage $t \leq$ 0.8 the linear stage of the disturbances development takes place when only their amplitudes increase without changing a shape.

At the second stage from 0.8 up to t = 1.2 one can observe some quasilinear stage of the disturbances development. Here the amplitudes also increase exponentially, a shape of the vortex sheet begins to change. Finally, at t = 1.2 the growth of the shear layer thickness ceases to be exponential and becomes linear, a nonlinear stage of the disturbances development starts. At this moment the vortex sheet "breaks and the amplitude of the transverse velocity reaches its maximum value.

We have carried out also the computations of the vortex sheet stability with respect to disturbances of its boundary. A value of the amplitude of the function h disturbance was varied within a range from 10^{-5} up to 10^{-1}. In these cases a character of the disturbances development turns out to be qualitatively the same under the action of acoustic disturbances. The presence of the linear stage of their development is also observed at the initial small amplitudes of disturbances. It should be noted that with increasing the initial amplitude of disturbances the duration of this stage decreases. At last, at initial disturbances of the order 0.01 the linear stage of the disturbances development is almost absent. Increasing more and more the initial amplitudes we can achieve the state when the disturbances will increase nonlinearly at once. At that the amplitude of disturbances of the vortex sheet boundary grows slowly (Pig.3a), and the disturbances of the velocity at the initial stage drastically increase and then fluctuate near some mean value (Fig. 3).

Up to now we have examined the evolution of the vortex sheet assuming the fluid inviscid. As a real fluid always possesses a finite viscosit3--, this fact should be taken into account when solving the problem. In the given problem the viscosity can be taken into consideration in a similar way as it has been suggested

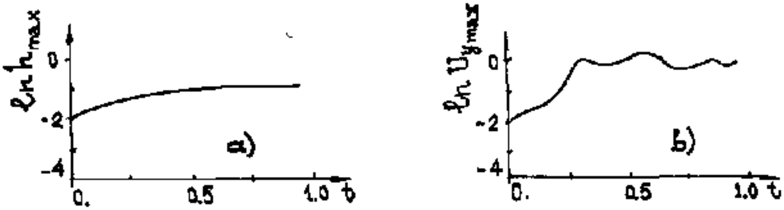


Fig.3. Rate of the increase of the functions h_{max} and $\vartheta_{y\,max}$

in our work [4], having implemented the splitting up of the equation of vorticity in terms of the physical processes.

It is well known that the vortex sheet in viscid fluid is unstable with respect to any disturbances. The regard for viscosity exerts a stabilizing influence. The critical Reynolds number which determines the beginning of the vortex sheet instability in the external acoustic field RG is equal to 65, This number decreases when the amplitude of the disturbances increases.

We have carried out similar computations as for the vortex sheet for a layer of finite thickness. Since the value of acoustic disturbances is small and their role is in fact reduced to excitation of vortex disturbances, the further development of which weakly depends on the presence of acoustic field, the computation results have proved to be in a good agreement with linear theory. For the vortex layer with a piecewise-linear velocity distribution the instability was observed at $k_x\Delta < 1.3$, and for a layer with velocity distribution according to the tangent law at $k_x\Delta < 1.$,

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